Matching Through the Sketching Lens: Alice (and Bob) in Combinatorial Optimization

Sudipto Guha

Work with Kook Jin Ahn. Arxiv 1307.4359

A Basic Question on Graphs: Maximum Matching

One of the first combinatorial algorithms, Edmonds '65.
Weighted graph $G = (V, E, w)$, find the maximum matching.

$|V| = n, |E| = m$. Possibly Nonbipartite. w_{ij} arbitrary.
Think $m \gg n$.

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$\epsilon > 0$ is a constant.
What is space is small? Mapreduce in a few rounds?

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More generally: Optimization over very large graphs?

How to analyse/define it?

Augmentation paths is a not a good idea ...

1. Find an initial solution. .
2. For $O(p/\epsilon)$ steps:
   2.1 Sample $n^{1+1/p}$ edges using current prices.
   2.2 Find the best weighted matching in the sample.
   2.3 Maintain the best weight matching found (say β) so far.
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2. Simulate t steps of a primal-dual algorithm trying to prove the dual $\leq \beta(1+O(\epsilon))$.
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This Talk: The Overall Goal

Takeaway 1. $(1-\epsilon)$-approximate Maximum Weighted Matching
Arbitrary order, find actual edges in a feasible integral solution.

$O(n^{1+1/p})$ space, $O(p/\epsilon)$ rounds of adaptive sketching/sampling.
Sampling weight of edge depends on $O(\mathrm{poly}(\log n, \frac{1}{\epsilon}))$ vertices
Running time $O(m\ \mathrm{poly}(\log n, \frac{1}{\epsilon}))$.

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- Weights, nonbipartite: dependence on n, time $n^{O(1/\epsilon)}$ etc.

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Takeaway 2. A Dual-Primal Strategy:
Solve the dual using sparsifiers of the weights on constraints.
(Note: these weights are now primal assignments.)
Prove: No progress (for the dual) on specific sparsifier $\implies$
That sparsifier contains a good primal solution.

Not true about arbitrary sparsifiers.

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$O(1/\epsilon)$ iterations using $O(n^{1.1})$ space.
Second Order: $\Omega(\epsilon^2 + \log n)$.
Best **First Order**: Bienstock Iyengar, $O\left(\frac{\sqrt{|\text{variables}|}}{\epsilon}\right)$.

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$\#$ of truly adaptive steps to solve an LP.
$\#$ of truly adaptive steps to solve a Matching LP.
The standard relaxation does not work.

The Standard Matching Polytope

$$\beta^* = \max \sum_{(i,j)} w_{ij} y_{ij}$$
$$\sum_j y_{ij} \le 1$$
$$\sum_{i,j\in U} y_{ij} \le \left\lfloor \frac{|U|}{2} \right\rfloor \quad \forall U$$
$$y_{ij} \ge 0$$

$$\beta^* = \min \sum_i x_i + \sum_U \left\lfloor \frac{|U|}{2} \right\rfloor z_U$$
$$x_i + x_j + \sum_{i,j\in U} z_U \ge w_{ij} \qquad \forall (i,j) \in E$$
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Forget space requirements for now...

Primal-dual & Problems ...

Packing: $O(n^{1/\epsilon})$ constraints, no structure

Covering:

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$$\textstyle\sum_{i\in U} \left(\sum_j y_{ij}\right) - \sum_{i\in U, j\notin U} y_{ij} \leq |U| - 1 \text{ for } |U| \text{ odd.}$$

Constraint on dual weights = affine combination of cuts!

Cut-Sparsifiers exists. Maybe we can sparsify dual weights?

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Primal-dual & Problems ...

Packing: $O(n^{1/\epsilon})$ constraints, no structure , width..

Covering: width (sensitivity) ..

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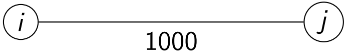

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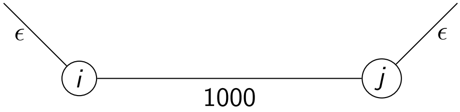

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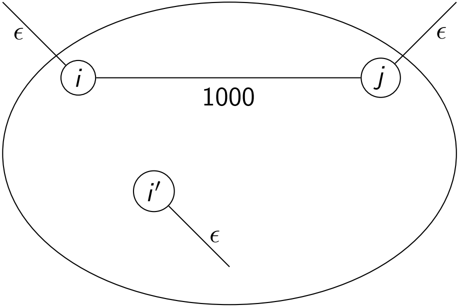

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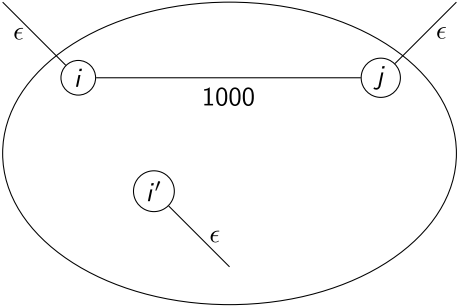

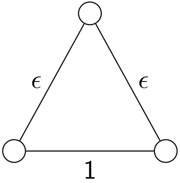

A New LP: Stage 1

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$$\hat{\beta} = \max \sum_{(i,j)} y_{ij} - 3 \sum_i \mu_i$$
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$$2x_i + \sum_{U : i \in U} z_U \le 3$$
$$x_i, z_U \ge 0$$

A New LP: Stage 2

Weighted Non-bipartite graphs. Assume weights are of form $w_k = (1+\epsilon)^k$, **not integers**.

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$$\hat{\beta} = \max \sum_k \hat{w}_k \left(\sum_{(i,j)\in \hat{E}_k} y_{ij} - 3 \sum_i \mu_{ik} \right)$$
$$\sum_{(i,j)\in \hat{E}_k} (y_{ij} - 2\mu_{ik}) \le y_{i(k)} \qquad \forall i, k$$
$$\sum_k y_{i(k)} \le 1 \qquad \forall i$$
$$\sum_{k \ge \ell} \left(\sum_{(i,j)\in \hat{E}_k, i,j \in U} y_{ij} - \sum_{i \in U} \mu_{ik} \right) \le \left\lfloor \frac{|U|}{2} \right\rfloor \qquad \forall U, \ell$$
$$y_{ij}, y_{i(k)}, \mu_{ik} \ge 0$$

The Dual-Primal Strategy (max version)

To solve [$*$] $\beta^* = \max \mathbf{c}^T\mathbf{y}; \mathcal{A}\mathbf{y} \leq \mathcal{b}$ subject to $\mathbf{y} \geq 0$.

Solve [**] $\quad \mathbf{Q}[\beta] : \begin{cases} \mathbf{u}_s^T \mathbf{A}^T \mathbf{x} \geq (1 - O(\epsilon))\mathbf{u}_s^T \mathbf{c} \\ \mathbb{G}(\mathbf{u}_s, \mathbf{x}) \\ \mathbf{x} \in \mathbf{Q}; \mathbf{b}^T \mathbf{x} \leq \beta \end{cases}$

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To solve [$*$] $\beta^* = \max \mathbf{c}^T\mathbf{y}; \mathcal{A}\mathbf{y} \leq \mathscr{b}$ subject to $\mathbf{y} \geq 0$.

$$\text{Solve [**]} \begin{cases} \mathbf{Q}[\beta] : \begin{cases} \mathbf{u}_s^T \mathbf{A}^T \mathbf{x} \geq (1 - O(\epsilon))\mathbf{u}_s^T \mathbf{c} \\ \mathbb{G}(\mathbf{u}_s, \mathbf{x}) \\ \mathbf{x} \in \mathbf{Q}; \mathbf{b}^T \mathbf{x} \leq \beta \end{cases} \\ \mathbf{P}\mathbf{x} \leq 2\mathbf{q} \text{ (the width } \implies \rho \text{ is small)} \end{cases}$$

The Dual-Primal Strategy (max version)

To solve [$*$] $\beta^* = \max \mathbf{c}^T\mathbf{y}; \mathcal{A}\mathbf{y} \leq \mathscr{b}$ subject to $\mathbf{y} \geq 0$.

$$\text{Solve [**]} \begin{cases} \mathbf{Q}[\beta] : \begin{cases} \mathbf{u}_s^T \mathbf{A}^T \mathbf{x} \geq (1 - O(\epsilon))\mathbf{u}_s^T \mathbf{c} \\ \mathbb{G}(\mathbf{u}_s, \mathbf{x}) \\ \mathbf{x} \in \mathbf{Q}; \mathbf{b}^T\mathbf{x} \leq \beta \end{cases} \\ \mathbf{Px} \leq 2\mathbf{q} \text{ (the width } \implies \rho \text{ is small)} \end{cases}$$

Solve: $\mathbf{x} \in \mathbf{Q}[\beta], \mathbf{Px} \leq 2\mathbf{q}$ by solving

$$\begin{cases} \mathbf{u}_s^T \mathbf{A}^T \mathbf{x} - \varrho\zeta^T \mathbf{Px} \geq (1 - O(\epsilon))\mathbf{u}_s^T \mathbf{c} - \varrho\zeta^T \mathbf{q} \\ \mathbb{G}(\mathbf{u}_s, \mathbf{x}) \\ \mathbf{x} \in \mathbf{Q}; \mathbf{b}^T\mathbf{x} \leq \beta; \mathbf{P}_i\mathbf{x} \leq \mathbf{q}_i \implies \mathbf{Px} \leq \rho_i \mathbf{q} \end{cases}$$

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Solve: $\mathbf{x} \in \mathbf{Q}[\beta], \mathbf{Px} \leq 2\mathbf{q}$ by solving

$$\left\{ \begin{array}{l} \mathbf{u}_s^T\mathbf{A}^T\mathbf{x} - \varrho\zeta^T\mathbf{Px} \geq (1 - O(\epsilon))\mathbf{u}_s^T\mathbf{c} - \varrho\zeta^T\mathbf{q} \\ \mathbb{G}(\mathbf{u}_s, \mathbf{x}) \\ \mathbf{x} \in \mathbf{Q}; \mathbf{b}^T\mathbf{x} \leq \beta; \mathbf{P}_i\mathbf{x} \leq \mathbf{q}_i \implies \mathbf{Px} \leq \rho_i\mathbf{q} \end{array} \right.$$

Or provide a solution to [$*$] such that $\mathbf{y}(i) > 0 \implies \mathbf{u}_s(i) > 0$

The Dual-Primal Strategy (max version)

To solve $[*]$ $\beta^* = \max \mathbf{c}^T\mathbf{y}; \mathcal{A}\mathbf{y} \le \mathcal{b}$ subject to $\mathbf{y} \ge 0$.

$$\text{Solve [**]} \begin{cases} \mathbf{Q}[\beta] : \begin{cases} \mathbf{u}_s^T \mathbf{A}^T \mathbf{x} \ge (1 - O(\epsilon))\mathbf{u}_s^T \mathbf{c} \\ \mathbb{G}(\mathbf{u}_s, \mathbf{x}) \\ \mathbf{x} \in \mathbf{Q}; \mathbf{b}^T\mathbf{x} \le \beta \end{cases} \\ \mathbf{P}\mathbf{x} \le 2\mathbf{q} \text{ (the width } \implies \rho \text{ is small)} \end{cases}$$

Solve: $\mathbf{x} \in \mathbf{Q}[\beta], \mathbf{P}\mathbf{x} \le 2\mathbf{q}$ by solving

$$\begin{cases} \mathbf{u}_s^T \mathbf{A}^T \mathbf{x} - \varrho\zeta^T\mathbf{P}\mathbf{x} \ge (1 - O(\epsilon))\mathbf{u}_s^T \mathbf{c} - \varrho\zeta^T\mathbf{q} \\ \mathbb{G}(\mathbf{u}_s, \mathbf{x}) \\ \mathbf{x} \in \mathbf{Q}; \mathbf{b}^T\mathbf{x} \le \beta; \mathbf{P}_i\mathbf{x} \le \mathbf{q}_i \implies \mathbf{P}\mathbf{x} \le \rho_i\mathbf{q} \end{cases}$$

Or provide a solution to $[*]$ such that $\mathbf{y}(i) > 0 \implies \mathbf{u}_s(i) > 0$

Clearly $\mathcal{A}, \mathbf{A}$ are related. But it need not be simple.
In particular $[**] \implies \beta^* \le (1 + O(\epsilon))\beta$

$(1-\epsilon)$ Approximate Maximum Weighted Matching

1. Find an initial solution. **Note:** of the dual. Nontrivial.
2. For $O(p/\epsilon)$ steps:
   2.1 **Sample** $n^{1+1/p}$ edges using current prices.
   2.2 Find the best weighted matching in the sample.
   2.3 Maintain the best weight matching found (say β) so far.
   2.4 **Update** the prices.

Update $\Longrightarrow$
1. Subdivide edges into $t = O(\frac{1}{p\epsilon}\log n)$ blocks
2. Simulate t steps of a primal-dual algorithm trying to prove the dual $\leq \beta(1+O(\epsilon))$.
3. Obtain new dual prices (of the dual).

Deferred Sparsifiers

Edge has a promise $\sigma_{ij}\frac{1}{\nu} \leq u_{ij} \leq \sigma_{ij}\nu$.

See the edges. Construct a data structure.

Then u_{ij} are known.

Construct a sparsifier preserving all cuts.

Needs a good oracle.

The Oracle: Surround the city by wood

Given $\mathbf{u}^s, \zeta$ solve:

$$\sum_{i,k} x_{i(k)} \left(\sum_{j:(i,j)\in \hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} \right) + \sum_{U\in\mathcal{O}_s,\ell} z_{U,\ell} \left(\sum_{k\geq \ell} \left(\sum_{(i,j)\in \hat{E}_k, i,j\in U} u^s_{ijk} - \varrho \sum_{i\in U} \zeta_{ik} \right) \right)$$

$$\geq (1 - O(\epsilon)) \sum_k \hat{w}_k \left(\sum_{(i,j)\in \hat{E}_k} u^s_{ijk} - 3\varrho \sum_i \zeta_{ik} \right)$$

$$2x_{i(k)} + \sum_{\ell\leq k} \left(\sum_{U\in\mathcal{O}_s, i\in U} z_{U,\ell} \right) \leq \left(\frac{24}{\epsilon} + \frac{24}{\epsilon^2} \right) \hat{w}_k \quad \forall i,k$$

$$z_{U,\ell} \left(\sum_{k\leq \ell} \left(\sum_{(i,j)\in \hat{E}_k, i,j\in U} u^s_{ijk} - \sum_{i\in U} \left(\sum_{j\notin U,(i,j)\in \hat{E}_k} u^s_{ijk} \right) \right) \right) \geq 0 \quad \forall U,\ell$$

$$x_i - x_{i(k)} \geq 0 \quad \forall i,k$$

$$\sum_i x_i + \sum_{\ell, U\in\mathcal{O}_s} z_{U,\ell} \left\lfloor \frac{|U|}{2} \right\rfloor \leq \beta$$

$$x_i, x_{i(k)}, z_{U,\ell} \geq 0$$

Consider the bipartite case ...

The Oracle: Part 3.1

$$\sum_{i,k} x_{i(k)} \left(\sum_{j:(i,j)\in \hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} \right) \geq \sum_k \hat{w}_k \left(\sum_{(i,j)\in \hat{E}_k} u^s_{ijk} - 3\varrho \sum_i \zeta_{ik} \right) = \gamma$$

$$2x_{i(k)} \leq \frac{24}{\epsilon} \hat{w}_k \quad \forall i, k$$

$$x_i - x_{i(k)} \geq 0 \quad \forall i, k$$

$$\sum_i x_i \leq \beta$$

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$$\sum_i x_i \leq \beta$$

Let $R = \{[i,k]\}$ s.t.

$$\sum_{j:(i,j)\in \hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} > 0.$$

For each i determine the largest k_i^* s.t.,

$$\sum_{[i,k]\in R, k>k_i^*} \left(\sum_{j:(i,j)\in \hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} \right) \leq \gamma/\beta$$

The Oracle: Part 3.1

$$\sum_{i,k} x_{i(k)} \left(\sum_{j:(i,j)\in\hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} \right) \geq \sum_k \hat{w}_k \left(\sum_{(i,j)\in\hat{E}_k} u^s_{ijk} - 3\varrho \sum_i \zeta_{ik} \right) = \gamma$$

$$2x_{i(k)} \leq \frac{24}{\epsilon}\hat{w}_k \quad \forall i,k$$

$$x_i - x_{i(k)} \geq 0 \quad \forall i,k$$

$$\sum_i x_i \leq \beta$$

$$\boxed{\begin{aligned} &\sum_{(i,j)\in\hat{E}_k} (y_{ij} - 2\mu_{ik}) \leq y_{i(k)} \quad \forall i,k \\ &\sum_k y_{i(k)} \leq 1 \quad \forall i \end{aligned}}$$

Let $R = \{[i,k]\}$ s.t.
$\sum_{j:(i,j)\in\hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} > 0.$

For each i determine the largest k_i^* s.t.,

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The Oracle: Part 3.2

$$\sum_{i,k} x_{i(k)} \left(\sum_{j:(i,j)\in \hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} \right) \geq \sum_k \hat{w}_k \left(\sum_{(i,j)\in \hat{E}_k} u^s_{ijk} - 3\varrho \sum_i \zeta_{ik} \right) = \gamma$$

$$2x_{i(k)} \leq \frac{24}{\epsilon} \hat{w}_k \quad \forall i, k$$

$$x_i - x_{i(k)} \geq 0 \quad \forall i, k$$

$$\sum_i x_i \leq \beta$$

Let $R = \{[i,k]\}$ s.t. $\displaystyle\sum_{j:(i,j)\in \hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} > 0$.

$$\sum_{[i,k]\in R, k > k_i^*} w_{k_i^*} \left(\sum_{j:(i,j)\in \hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} \right) +$$

$$\sum_{[i,k]\in R, k \leq k_i^*} w_k \left(\sum_{j:(i,j)\in \hat{E}_k} u^s_{ijk} - 2\varrho\zeta_{ik} \right) \text{ versus } \epsilon\gamma/(12\beta).$$

In the future

Your favorite combinatorial optimization problem.

A new perspective ...

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That's all folks.